

# Decentralized Restless Bandit with Multiple Players and Unknown Dynamics

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**Abstract**—We consider decentralized restless multi-armed bandit problems with unknown dynamics and multiple players. The reward state of each arm transits according to an unknown Markovian rule when it is played and evolves according to an arbitrary unknown random process when it is passive. Players activating the same arm at the same time collide and suffer from reward loss. The objective is to maximize the long-term reward by designing a decentralized arm selection policy to address unknown reward models and collisions among players. A decentralized policy is constructed that achieves a regret with logarithmic order when an arbitrary nontrivial bound on certain system parameters is known. When no knowledge about the system is available, we extend the policy to achieve a regret arbitrarily close to the logarithmic order. The result finds applications in communication networks, financial investment, and industrial engineering.

## I. INTRODUCTION

### A. The Classic MAB with A Single Player

In the classic MAB, there are  $N$  independent arms and a single player. Each arm, when played, offers an i.i.d. random reward to the player. The reward distribution of each arm is unknown. At each time, the player chooses one arm to play, aiming to maximize the total expected reward in the long run. This problem involves the well-known tradeoff between exploitation and exploration. For exploitation, the player should select the arm with the largest sample mean of reward. For exploration, the player should select an underplayed arm to learn its reward statistics.

Under the non-Bayesian formulation, the performance measure of an arm selection policy is the so-called *regret* or *the cost of learning* defined as the reward loss with respect to the case with known reward models [1]. In 1985 Lai and Robbins showed that the minimum regret grows at a logarithmic order under certain regularity conditions [1]. The best leading constant was also obtained, and an optimal policy was constructed to achieve the minimum regret growth rate (both the logarithmic order and the best leading constant). In 1987, Anantharam *et al.* extended Lai and Robbins's results to accommodate multiple simultaneous plays [2] and a Markovian reward model where the reward of each arm evolves as an unknown Markov process over successive plays

and remains frozen when the arm is passive (the so-called *rested* Markovian reward model) [3].

Several other simpler policies have been developed to achieve logarithmic regret for the classic MAB under an i.i.d. reward model [4], [5]. In particular, the index policy—referred to as Upper Confidence Bound 1 (UCB-1)—proposed in [5] achieves the logarithmic regret with a uniform bound on the leading constant over time. In [6], UCB-1 was extended to the rested Markovian reward model adopted in [3].

### B. Decentralized MAB with Distributed Multiple Players

In [7], Liu and Zhao formulated and studied a decentralized version of the classic MAB with  $M$  ( $M < N$ ) distributed players under the i.i.d. reward model. Different arms can have different reward distributions and they are unknown to the players. At each time, a player chooses one arm to play based on its *local* observation and decision history without exchanging information with other players. Collisions occur when multiple players choose the same arm, and, depending on the collision model, either no one receives reward or the colliding players share the reward in an arbitrary way. The objective is to maximize the long-term sum reward from all players. Another desired feature of policies for decentralized MAB is fairness, *i.e.*, different players have the same expected reward growth rate. Liu and Zhao proposed the Time Division Fair Sharing (TDFS) framework, it achieves the same logarithmic regret order as the centralized case where all players share their observations in learning and collisions are eliminated through centralized perfect scheduling [7]. Assuming a Bernoulli reward model, decentralized MAB was also addressed in [8], where the single-player policy UCB-1 was extended to the multi-player setting.

### C. Main Results

In this paper, we consider the decentralized MAB with a restless Markovian reward model. In a single-player restless MAB, the reward state of each arm transits according to an unknown Markovian rule when played and transits according to an arbitrary unknown random process when passive as addressed in our prior work [9]. In [9], we proposed a policy Restless UCB (RUCB), which achieves a logarithmic order of the weak regret defined as the reward loss compared to the case when the player knows which arm is the most rewarding and always plays the best arm. RUCB borrows the index form

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of UCB-1 given in [5] and has a deterministic epoch structure with carefully chosen epoch lengths to balance exploration and exploitation. The concept of weak regret was first used in [10]; it measures the reward loss with respect to the optimal *single-arm* policy, which, while optimal under the i.i.d. and rested Markovian reward models (up to an  $O(1)$  term of loss for the latter), is no longer optimal in general under a known restless reward model. Analysis of the strict regret of restless MAB is in general intractable given that finding the optimal policy of a restless bandit under *known* model is itself PSPACE-hard in general [11].

In this paper, we extend RUCB proposed in our prior work [9] to a decentralized setting of restless MAB with multiple players. We consider two types of restless reward models: exogenous restless model and endogenous restless model. In the former, the system itself is rested: the state of an arm does not change when the arm is not engaged. However, from each individual player's perspective, arms are restless due to actions of other players that are unobservable and uncontrollable. Under the endogenous restless model, the state of an arm evolves according to an arbitrary unknown random process even when the arm is not played. Under both restless models, we extend RUCB to achieve a logarithmic order of the regret. The result for the exogenous restless model, however, is stronger in the sense that the regret is indeed defined with respect to the optimal policy under known reward models. This is possible due to the inherent *rested* nature of the systems.

There are a couple of parallel work to [9] on the single-player restless MAB. In [12], Tekin and Liu adopted the weak regret and proposed a policy that achieves logarithmic (weak) regret when certain knowledge about the system parameters is available [12]. The policy proposed in [12] also uses the index form of UCB-1 given in [5], but the structure is different from RUCB proposed in [9]. Specifically, under the policy proposed in [12], an arm is played consecutively for a random number of times determined by the regenerative cycle of a particular state, and observations obtained outside the regenerative cycle are not used in learning. RUCB, however, has a deterministic epoch structure, and all observations are used in learning. In [13], the strict regret was considered for a special class of restless MAB. Specifically, when arms are governed by stochastically identical two-state Markov chains, a policy was constructed in [13] to achieve a regret with an order arbitrarily close to logarithmic.

**Notation** For two positive integers  $k$  and  $l$ , define  $k \oslash l \triangleq ((k-1) \bmod l) + 1$ , which is an integer taking values from  $1, 2, \dots, l$ .

## II. PROBLEM FORMULATION

In the decentralized MAB problem, we have  $M$  players and  $N$  independent arms. At each time, each player chooses one arm to play. Each arm, when played (activated), offers certain amount of reward that models the current state of the arm. Let  $s_j(t)$  and  $\mathcal{S}_j$  denote the state of arm  $j$  at time  $t$  and the state space of arm  $j$  respectively. Different arms can have different state spaces. When arm  $j$  is played, its state changes

according to a Markovian rule with  $P_j$  as the transition matrix. The transition matrixes are assumed to be irreducible, aperiodic, and reversible. As for the state transition of passive arms, we consider two models: endogenous restless model and exogenous restless model. In the endogenous restless model, arm states change in arbitrary ways when not played. In the exogenous restless model, arm states remain frozen if not engaged. The players do not know the transition matrices of the arms and do not communicate with each other. Conflicts occur when different players select the same arm to play. Under different conflict models, either the players in conflict share the reward or no one obtains any reward. The objective is to maximize the expected total reward collected in the long run. Let  $\vec{\pi}_j = \{\pi_s^j\}_{s \in \mathcal{S}_j}$  denote the stationary distribution of arm  $j$  (under  $P_j$ ), where  $\pi_s^j$  is the stationary probability (under  $P_j$ ) that arm  $j$  is in state  $s$ . The stationary mean reward  $\mu_j$  is given by  $\mu_j = \sum_{s \in \mathcal{S}_j} s \pi_s^j$ . Let  $\sigma$  be a permutation of  $\{1, \dots, N\}$  such that

$$\mu_{\sigma(1)} \geq \mu_{\sigma(2)} \geq \mu_{\sigma(3)} \geq \dots \geq \mu_{\sigma(N)}.$$

A policy  $\Phi$  is a rule that specifies an arm to play based on the local observation history. Let  $t_j(n)$  denote the time index of the  $n$ th play on arm  $j$ , and  $T_j(t)$  the total number of plays on arm  $j$  by time  $t$ . Notice that both  $t_j(n)$  and  $T_j(t)$  are random variables with distributions determined by the policy  $\Phi$ . Under the conflict model where players in conflict share the reward, the total reward by time  $t$  is given by

$$R(t) = \sum_{j=1}^N \sum_{n=1}^{T_j(t)} s_j(t_j(n)). \quad (1)$$

Under the conflict model where no players in conflict obtain any reward, the total reward by time  $t$  is given by

$$R(t) = \sum_{j=1}^N \sum_{n=1}^{T_j(t)} s_j(t_j(n)) \mathbb{I}_j(t_j(n)). \quad (2)$$

where  $\mathbb{I}_j(t_j(n)) = 1$  if arm  $j$  is played by one and only one player at time  $t_j(n)$ , and  $\mathbb{I}_j(t_j(n)) = 0$  otherwise.

As mentioned in Sec. I, for both restless models, performance of any policy  $\Phi$  is evaluated using regret  $r_\Phi(t)$  defined as the reward loss with respect to having  $M$  best arms constantly engaged. Specifically, for both restless models, regret is defined as follows:

$$r_\Phi(t) = t \sum_{i=1}^M \mu_{\sigma(i)} - \mathbb{E}_\Phi R(t) + O(1), \quad (3)$$

where the constant  $O(1)$  is caused by the transient effects of playing the  $M$  best arms,  $\mathbb{E}_\Phi$  denotes the expectation with respect to the random process induced by policy  $\Phi$ . The objective is to minimize the growth rate of the regret. Note that the constant  $O(1)$  term can be ignored when studying the growth rate of the regret.

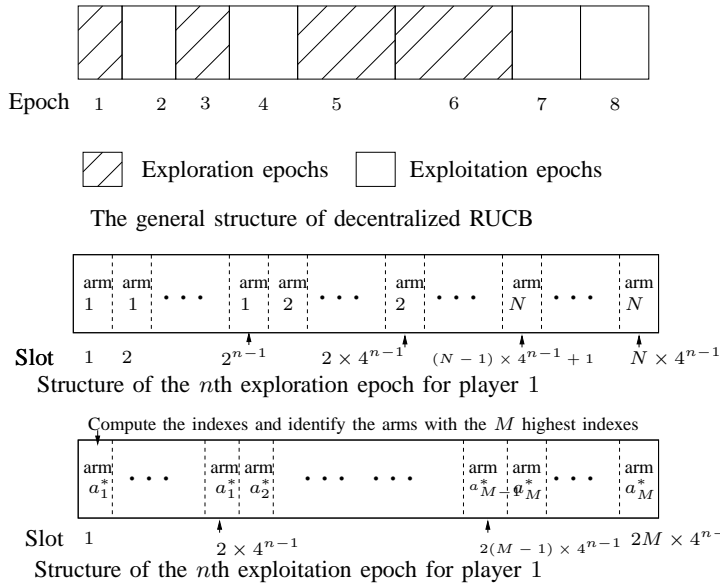


Fig. 1. Epoch structures of decentralized RUCB

### III. THE DECENTRALIZED RUCB POLICY

The proposed decentralized RUCB is based on an epoch structure. We divide the time into disjoint epochs. There are two types of epochs: exploitation epochs and exploration epochs (see an illustration in Fig. 1). In the exploitation epochs, the players calculate the indexes of all arms and play the arms with the  $M$  highest indexes, which are believed to be the  $M$  best arms. In the exploration epochs, the players obtain information of all arms by playing them equally many times. The purpose of the exploration epochs is to make decisions in the exploitation epochs sufficiently accurate. As shown in Fig. 1, in the  $n$ th exploration epoch, each player plays every arm  $4^{n-1}$  times. At the beginning of the  $n$ th exploitation epoch the player calculates index for every arm (see (5) in Fig. 2) and selects the arm with the  $M$  highest indexes (denoted as arm  $a_{(1)}^*$  to arm  $a_{(M)}^*$ ). Each exploitation epoch is divided into  $M$  subepochs with each having a length of  $2 \times 4^{n-1}$ . Player  $k$  plays arm  $a_{((m-k+M+1) \odot M)}^*$  in the  $m$ th subepoch of each exploitation epoch. The details on interleaving the two types of epochs are given in Step 2 in Fig. 2. Specifically, whenever sufficiently many ( $D \ln t$ , see (4)) observations have been obtained from every arm in the exploration epochs, the player is ready to proceed with a new exploitation epoch. Otherwise, another exploration epoch is required to gain more information about each arm. It is also implied in (4) that only logarithmically many plays are spent in the exploration epochs, which is one of the key reasons for the logarithmic regret of decentralized RUCB. This also implies that the exploration epochs are much less frequent than the exploitation epochs. Though the exploration epochs can be understood as the “information gathering” phase, and the exploitation epochs as the “information utilization” phase, observations obtained in the exploitation epochs are also used in learning the arm dynamics. This can be seen in Step 3 in

Fig. 2. The epoch structure, (*i.e.*, the starting and ending points of epochs) are prefixed numbers only depending on parameter  $D$ . This is one of the key reasons why different players can be coordinated (*i.e.*, entering the same epoch at the same time) without intercommunications.

#### Decentralized RUCB

Time is divided into epochs. There are two types of epochs, exploration epochs and exploitation epochs. At the beginning of the  $n$ th exploitation epoch, we choose the  $M$  arms to play, each of them for  $2 \times 4^{n-1}$  many times. In the  $n$ th exploration epoch, we play every arm  $4^{n-1}$  many times. Let  $n_O(t)$  denote the number of exploration epochs played by time  $t$  and  $n_I(t)$  the number of exploitation epochs played by time  $t$ .

1. At  $t = 1$ , we start the first exploration epoch, in which every arm is played once. We set  $n_O(N+1) = 1$ ,  $n_I(N+1) = 0$ . Then go to Step 2.
2. Let  $X_1(t) = (4^{n_O(t)} - 1)/3$  be the time spent on each arm in exploration epochs by time  $t$ . Choose  $D$  according to (6)(7). If

$$X_1(t) > D \ln t, \quad (4)$$

go to Step 3 (start an exploitation epoch). Otherwise, go to Step 4 (start an exploration epoch).

3. Calculate indexes  $d_{i,t}$  for all arms using the formula below:

$$d_{i,t} = \bar{s}_i(t) + \sqrt{\frac{L \ln t}{T_i(t)}}, \quad (5)$$

where  $t$  is the current time,  $\bar{s}_i(t)$  is the sample mean from arm  $i$  by time  $t$ ,  $L$  is chosen according to (6), and  $T_i(t)$  is the number of times we have played arm  $i$  by time  $t$ . Then choose the arms with the  $M$  highest indexes (arm  $a_{(1)}^*$  to arm  $a_{(M)}^*$ ). Each exploitation epoch is divided into  $M$  subepochs with each having a length of  $2 \times 4^{n-1}$ . Player  $k$  plays arm  $a_{((m-k+M+1) \odot M)}^*$  in the  $m$ th subepoch of each exploitation epoch. After arm  $a_{(1)}^*$  to arm  $a_{(M)}^*$  are played, increase  $n_I$  by one and go to step 2.

4. Each Play each arm for  $4^{(n_O-1)}$  slots. Each exploration epoch is divided into  $N$  subepochs with each having a length of  $4^{(n_O-1)}$ . Player  $k$  plays arm  $a_{(m-k+N+1 \odot N)}^*$  in the  $m$ th subepoch of each exploitation epoch. After all the arms are played, increase  $n_I$  by one and go to step 2.

Fig. 2. Decentralized RUCB policy

#### A. Eliminate Pre-Agreement

So far we have assumed a pre-agreement among the players: they target at the  $M$  best arms with different offsets to avoid excessive collisions. In this subsection, we show that this pre-agreement can be eliminated while maintaining the logarithmic order of the system regret. Furthermore, players can join the system at different times without any global synchronization. Specifically, at each player, the structure of the exploration and exploitation epochs is the same as the local RUCB policy with pre-agreement. The only difference here is that in each exploitation epoch, the player randomly chooses one of the  $M$  arms considered as the best to play whenever a collision with other players is observed. If no collision is observed, the player keeps playing the same arm. This simple elimination

of pre-agreement leads to a complete decentralization among players while achieving the same logarithmic order of the system regret. Except that each player can join the system according to the local schedule, the player can also leave the system for an arbitrary finite time period.

#### IV. THE LOGARITHMIC REGRET OF DECENTRALIZED RUCB

In this section, we show that the regret achieved by the decentralized RUCB policy has a logarithmic order. This is given in the following theorem.

*Theorem 1:* Under the exogenous restless Markovian reward model, assume that when arms are engaged, they can be modeled as finite state, irreducible, aperiodic, and reversible Markov chains. All the states (rewards) are positive. Let  $\pi_{\min} = \min_{s \in \mathcal{S}_i, 1 \leq i \leq N} \pi_s^i$ ,  $\epsilon_{\max} = \max_{1 \leq i \leq N} \epsilon_i$ ,  $\epsilon_{\min} = \min_{1 \leq i \leq N} \epsilon_i$ ,  $s_{\max} = \max_{s \in \mathcal{S}_i, 1 \leq i \leq N} s$ ,  $s_{\min} = \min_{s \in \mathcal{S}_i, 1 \leq i \leq N} s$ , and  $|\mathcal{S}|_{\max} = \max_{1 \leq i \leq N} |\mathcal{S}_i|$  where  $\epsilon_i = 1 - \lambda_i$  ( $\lambda_i$  is the second largest eigenvalue of the matrix  $P_i$ ). Assume that different arms have different  $\mu$  values<sup>1</sup> Set the policy parameters  $L$  and  $D$  to satisfy the following conditions:

$$L \geq \frac{1}{\epsilon_{\min}} \left( 4 \frac{20s_{\max}^2 |\mathcal{S}|_{\max}^2}{(3 - 2\sqrt{2})} + 10s_{\max}^2 \right), \quad (6)$$

$$D \geq \frac{4L}{(\min_{j \leq M} (\mu_{\sigma(j)} - \mu_{\sigma(j+1)}))^2}. \quad (7)$$

Under the conflict model where players share the reward, the regret of decentralized RUCB at the end of any epoch can be upper bounded by

$$\begin{aligned} r_{\Phi}(t) \leq & \frac{1}{3} [4(3D \ln t + 1) - 1] \left( \sum_{i=1}^M \mu_{\sigma(i)} - \frac{M}{N} \sum_{i=1}^N \mu_{\sigma(i)} \right) \\ & + 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\ & \sum_{i=1}^{M-1} \sum_{j=1, j \neq i}^N \mu_{\sigma(i)} \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\ & + 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\ & \sum_{j=M+1}^N (\mu_{\sigma(M)} - \mu_{\sigma(j)}) \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\ & + 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\ & \sum_{j=1}^{M-1} \mu_{\sigma(M)} \frac{|\mathcal{S}_{\sigma(M)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\ & + \sum_{i=1}^N \left[ \left( \min_{s \in \mathcal{S}_i} \pi_s \right)^{-1} \sum_{s \in \mathcal{S}_i} s \right] \end{aligned} \quad (8)$$

<sup>1</sup>This assumption can be relaxed by utilizing the shared index set. This assumption is only for simplicity of the presentation.

Under the model where no player in conflict gets any reward, the regret of decentralized RUCB at the end of any epoch can be upper bounded by:

$$\begin{aligned} r_{\Phi}(t) \leq & 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\ & \left( \sum_{i=1}^M \mu_{\sigma(i)} \right) \sum_{i=1}^M \sum_{j=1, j \neq i}^N \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\ & + \frac{1}{3} [4(3D \ln t + 1) - 1] \\ & \left( \sum_{i=1}^M \mu_{\sigma(i)} - \frac{M}{N} \sum_{i=1}^N \mu_{\sigma(i)} \right) \\ & + \sum_{i=1}^N \left[ \left( \min_{s \in \mathcal{S}_i} \pi_s \right)^{-1} \sum_{s \in \mathcal{S}_i} s \right] \end{aligned} \quad (9)$$

We point out that upper bounds of regret in Theorem 1 can be extended to any time  $t$  instead of only for ending points of epochs. They can also be extended to the endogenous restless model in terms of weak regret. The no pre-agreement version of decentralized RUCB can also achieve regret with a logarithmic order.

*Proof:* See Appendix A for details. ■

Theorem 1 requires an arbitrary (nontrivial) bound on  $s_{\max}^2$ ,  $|\mathcal{S}|_{\max}$ ,  $\epsilon_{\min}$ , and  $\min_{j \leq M} (\mu_{\sigma(j)} - \mu_{\sigma(j+1)})$ . In the case where these bounds are unavailable,  $D$  and  $L$  can be chosen to increase with time to achieve a regret order arbitrarily close to logarithmic order. This is formally stated in the following theorem.

*Theorem 2:* Assume the exogenous restless model and that all arms, when engaged, are modeled as finite state, irreducible, aperiodic, and reversible Markov chains. For any increasing sequence  $f(t)$  ( $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ), if  $L(t)$  and  $D(t)$  are chosen such that  $L(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ,  $\frac{f(t)}{D(t)} \rightarrow \infty$  as  $t \rightarrow \infty$ , and  $\frac{D(t)}{L(t)} \rightarrow \infty$  as  $t \rightarrow \infty$ , then we have

$$r_{\Phi}(t) \sim o(f(t) \log(t)). \quad (10)$$

We point out that the conclusion in Theorem 2 still holds for the endogenous restless model, though the proof needs to be modified.

*Proof:* See Appendix B for details. ■

#### V. CONCLUSION

In this paper, we studied the decentralized restless multi-armed bandit problems, where distributed players aim to accrue the maximum long-term reward without knowing the system reward statistics. Under the exogenous model where the arm reward status remains static when not engaged, we proposed a policy to achieve the optimal logarithmic order of the system regret. Under the endogenous model where the arm reward status evolves according to an arbitrary random process when not engaged, we showed that the proposed policy achieves a logarithmic (weak) regret. Furthermore, we showed that the proposed policy achieves a complete decentralization

where no pre-agreement or global synchronization among players is required.

#### APPENDIX A. PROOF OF THEOREM 1

We first rewrite the definition of regret as

$$\begin{aligned}
r_\Phi(t) &= t \sum_{i=1}^M \mu_{\sigma(i)} - \mathbb{E}_\Phi R(t) \\
&= \sum_{i=1}^N [\mu_i \mathbb{E}[T_i(t)] - \mathbb{E}[\sum_{n=1}^{T_i(t)} s_i(t_i(n))]] \\
&\quad + \mathbb{E}[\sum_{n=1}^{T_i(t)} s_i(t_i(n))] - \mathbb{E}_\Phi R(t) \\
&\quad + t \sum_{i=1}^M \mu_{\sigma(i)} - \sum_{i=1}^N \mu_i \mathbb{E}[T_i(t)]. \tag{11}
\end{aligned}$$

To bound the first term in (11), Lemma 1 is introduced below:

*Lemma 1* [3]: Let  $Y_1, Y_2, \dots$  be Markovian with state space  $\mathcal{S}$ , matrix of transition probabilities  $P$ , an initial distribution  $\vec{q}$ , and stationary distribution  $\vec{\pi}$  ( $\pi_s$  is the stationary probability of state  $s$ ). Let  $F_t$  be the  $\sigma$ -algebra generated by  $Y_1, Y_2, \dots, Y_t$  and  $G$  an  $\sigma$ -algebra independent of  $Y_\infty = \bigvee Y_t$ . Let  $T$  be a stopping time of  $\{F_t \vee G\}$ . The state (reward) at time  $t$  is denoted by  $s(t)$ . Let  $\mu$  denote the mean reward. For any stopping time  $T$ , there exists a value  $A_P \leq (\min_{s \in \mathcal{S}} \pi_s)^{-1} \sum_{s \in \mathcal{S}} s$  such that  $\mathbb{E}[\sum_{t=1}^T s(t) - \mu T] \leq A_P$ .

Using Lemma 1 the first term in (11) can be bounded by the following constant:

$$\sum_{i=1}^N [(\min_{s \in \mathcal{S}_i} \pi_s)^{-1} \sum_{s \in \mathcal{S}_i} s] \tag{12}$$

To show that the regret has a logarithmic order, it is sufficient to show that the second term plus the third term in (11) has a logarithmic order. These two terms can be understood as regret caused by two reasons. The first one is engaging bad arms in the exploration epochs. The second one is not playing the expected arms in the exploitation epochs. To show the second term in (11) has a logarithmic order, it is sufficient to show that the regret caused by the two reasons above have logarithmic orders.

Let  $\mathbb{E}[T_O(t)]$  denote the time spent on each arm in the exploration epochs by time  $t$  and an upper bound on  $T_O(t)$  is:

$$T_O(t) \leq \frac{1}{3} [4(3D \ln t + 1) - 1]. \tag{13}$$

Consequently the regret caused by engaging bad arms in the exploration epochs by time  $t$  is upper bounded by

$$\frac{1}{3} [4(3D \ln t + 1) - 1] \left( \sum_{i=1}^M \mu_{\sigma(i)} - \frac{M}{N} \sum_{i=1}^N \mu_{\sigma(i)} \right). \tag{14}$$

The second reason for regret in the second term of (11) is not playing the expected arms in the exploitation epochs. Let  $t_n$  denote the beginning point to the  $n$ th exploitation epoch. Let  $\Pr[i, j, n]$  denote the possibility that arm  $i$  has a higher index than arm  $j$  at  $t_n$ , where  $\mu_i < \mu_j$  and  $\mu_j \geq \mu_{\sigma(M)}$ . It can be shown that:

$$\Pr[i, j, n] \leq \frac{|\mathcal{S}_i| + |\mathcal{S}_j|}{\pi_{\min}} \left(1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}}\right) t_n^{-1} \tag{15}$$

Since different subepochs in the exploitation epochs are symmetric, the regret in different subepochs are the same. In the first subepoch, player  $k$  aims at arm  $\sigma(k)$ . In the model where players in conflict share the reward, player  $k$  failing to identify arm  $\sigma(k)$  in the first subepoch of the  $n$ th exploitation epoch can lead to a regret no more than  $\mu_{\sigma(k)} 2 \times 4^{n-1}$ . In calculating the upper bound for regret, for player  $M$ , we can assume that playing the arm  $\sigma(M+1)$  to arm  $\sigma(N)$  can contribute to the total reward. Thus an upper bound for regret in the  $n$ th exploitation epoch can be obtained as

$$\begin{aligned}
&2M4^{n-1} \left(1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}}\right) t_n^{-1} \left[ \sum_{i=1}^{M-1} \sum_{j=1, j \neq i}^N \mu_i \frac{|\mathcal{S}_i| + |\mathcal{S}_j|}{\pi_{\min}} \right. \\
&+ \sum_{j=1}^{M-1} \mu_M \frac{|\mathcal{S}_M| + |\mathcal{S}_j|}{\pi_{\min}} \\
&+ \left. \sum_{j=M+1}^N (\mu_M - \mu_j) \frac{|\mathcal{S}_i| + |\mathcal{S}_j|}{\pi_{\min}} \right] \tag{16}
\end{aligned}$$

By time  $t$ , at most  $(t - N)$  time slots have been spent on the exploitation epochs. Thus

$$n_I(t) \leq \lceil \log_4 \left( \frac{3}{2} (t - N) + 1 \right) \rceil. \tag{17}$$

From the upper bound on the number of the exploitation epochs given in (17), and also the fact that  $t_n \geq \frac{2}{3} 4^{n-1}$ , we have the following upper bound on regret caused in the exploitation epochs by time  $t$  (Denoted by  $r_{\Phi, I}(t)$ ):

$$\begin{aligned}
r_{\Phi, I}(t) &\leq 3 \lceil \log_4 \left( \frac{3}{2} (t - N) + 1 \right) \rceil \left(1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}}\right) \\
&\quad \sum_{i=1}^{M-1} \sum_{j=1, j \neq i}^N \mu_{\sigma(i)} \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\
&\quad + 3 \lceil \log_4 \left( \frac{3}{2} (t - N) + 1 \right) \rceil \left(1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}}\right) \\
&\quad \sum_{j=M+1}^N (\mu_{\sigma(M)} - \mu_{\sigma(j)}) \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\
&\quad + 3 \lceil \log_4 \left( \frac{3}{2} (t - N) + 1 \right) \rceil \left(1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}}\right) \\
&\quad \sum_{j=1}^{M-1} \mu_{\sigma(M)} \frac{|\mathcal{S}_{\sigma(M)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \tag{18}
\end{aligned}$$

Combining (11) (12) (14) (18), we can get the upper bound of regret:

$$\begin{aligned}
r_\Phi(t) \leq & \frac{1}{3}[4(3D \ln t + 1) - 1] \left( \sum_{i=1}^M \mu_{\sigma(i)} - \frac{M}{N} \sum_{i=1}^N \mu_{\sigma(i)} \right) \\
& + 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\
& \sum_{i=1}^{M-1} \sum_{j=1, j \neq i}^N \mu_{\sigma(i)} \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\
& + 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\
& \sum_{j=M+1}^N (\mu_{\sigma(M)} - \mu_{\sigma(j)}) \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\
& + 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\
& \sum_{j=1}^{M-1} \mu_{\sigma(M)} \frac{|\mathcal{S}_{\sigma(M)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\
& + \sum_{i=1}^N [(\min_{s \in \mathcal{S}_i} \pi_s)^{-1} \sum_{s \in \mathcal{S}_i} s] \quad (19)
\end{aligned}$$

Next we consider the model where no player in conflict gets reward. In the first subepoch of the  $n$ th exploitation epoch, each mistake by player  $k$  can cause regret more than  $\mu_{\sigma(k)} 2 \times 4^{n-1}$ . Assuming each mistake can cause  $\sum_{i=1}^M \mu_{\sigma(i)} 2 \times 4^{n-1}$  regret leads to the following upper bound for regret under this conflict model:

$$\begin{aligned}
r_\Phi(t) \leq & 3 \lceil \log_4 \left( \frac{3}{2}(t - N) + 1 \right) \rceil \left( 1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}} \right) \\
& \left( \sum_{i=1}^M \mu_{\sigma(i)} \right) \sum_{i=1}^M \sum_{j=1, j \neq i}^N \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}} \\
& + \frac{1}{3}[4(3D \ln t + 1) - 1] \\
& \left( \sum_{i=1}^M \mu_{\sigma(i)} - \frac{M}{N} \sum_{i=1}^N \mu_{\sigma(i)} \right) \\
& + \sum_{i=1}^N [(\min_{s \in \mathcal{S}_i} \pi_s)^{-1} \sum_{s \in \mathcal{S}_i} s] \quad (20)
\end{aligned}$$

## APPENDIX B. PROOF OF THEOREM 2

The choice of  $L(t)$  and  $D(t)$  implies that  $D(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . The regret has three parts: the transient effect of arms, the regret caused by playing bad arms in the exploration epochs, and the regret caused by mistakes in the exploitation epochs. It will be shown that each part of the regret is on a lower order than  $f(t) \log(t)$ . The transient effect of arms is the same as in Theorem 1. Thus it is upper bounded by a constant independent of time  $t$  and is on a lower order than  $f(t) \log(t)$ .

The regret caused by playing bad arms in the exploration epochs is bounded by

$$\frac{1}{3}[4(3D(t) \ln t + 1) - 1] \left( \sum_{i=1}^M \mu_{\sigma(i)} - \frac{M}{N} \sum_{i=1}^N \mu_{\sigma(i)} \right). \quad (21)$$

Since  $\frac{f(t)}{D(t)} \rightarrow \infty$  as  $t \rightarrow \infty$ , the part of regret in (21) is on a lower order than  $f(t) \log(t)$ .

For the regret caused by playing bad arms in the exploitation epochs, it is shown below that the time spent on a bad arm  $i$  can be bounded by a constant independent of  $t$ .

Since  $\frac{D(t)}{L(t)} \rightarrow \infty$  as  $t \rightarrow \infty$ , there exists a time  $t_1$  such that  $\forall t \geq t_1$ ,  $D(t) \geq \frac{4L(t)}{(\min_{j \leq M} (\mu_{\sigma(j)} - \mu_{\sigma(j+1)}))^2}$ . There also exists a time  $t_2$  such that  $\forall t \geq t_2$ ,  $L(t) \geq \frac{1}{\epsilon_{\min}} (7 \frac{20s_{\max}^2 |\mathcal{S}|_{\max}^2}{(3-2\sqrt{2})} + 10s_{\max}^2)$ . The time spent on playing bad arms before  $t_3 = \max(t_1, t_2)$  is at most  $t_3$ , and the caused regret is at most  $(\sum_{j=1}^M \mu_{\sigma(j)}) t_3$ . The regret caused by mistakes after  $t_3$  is upper bounded by  $6(1 + \frac{\epsilon_{\max} \sqrt{L}}{10s_{\min}}) (\sum_{i=1}^M \mu_{\sigma(i)}) \sum_{i=1}^M \sum_{j=1, j \neq i}^N \frac{|\mathcal{S}_{\sigma(i)}| + |\mathcal{S}_{\sigma(j)}|}{\pi_{\min}}$ . Thus the regret caused by mistakes in the exploitation epochs is on a lower order than  $f(t) \log(t)$ .

Because each part of the regret is on a lower order than  $f(t) \log(t)$ , the total regret is also on a lower order than  $f(t) \log(t)$ .

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